Published by Institute of Physics Publishing for SISSA

Received: *May 17, 2008*
Revised: *July 3, 2008*
Accepted: *July 8, 2008*
Published: *July 21, 2008*

Anomaly conditions for non-Abelian finite family symmetries

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Abstract: Assuming that finite family symmetries are gauged, we derive discrete anomaly conditions for various non-Abelian groups. We thus provide new constraints for flavor model building, in which discrete non-Abelian symmetries are employed to explain the tri-bimaximal mixing pattern in the lepton sector.

Keywords: Neutrino Physics, Discrete and Finite Symmetries.

JHEP07(2008)085

Contents

1. Introduction

Forty years after the setup of the Homestake experiment [1], the concept of neutrino oscillations is well established and generally accepted to explain the solar neutrino puzzle. Solar [2, 3], atmospheric [4], reactor [5], and accelerator [6] neutrino experiments all made important contributions to our current knowledge of the neutrino sector. In particular, one finds that the measured Maki-Nakagawa-Sakata-Pontecorvo (MNSP) leptonic mixing matrix approximately displays the so-called tri-bimaximal pattern [7, 8]. This fact has

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triggered an overwhelming interest in non-Abelian finite groups as means to explain the family structure of leptons and quarks.

Although differing in their details, all proposed models augment the Standard Model (SM) gauge symmetry with a discrete symmetry. If this discrete symmetry $\mathcal{G}$ originates in a continuous gauge symmetry G which is spontaneously broken, one refers to it as a *discrete gauge symmetry*. The assumption of a gauge origin has the advantage that the remnant discrete symmetry $\mathcal{G}$ is protected against violations by quantum gravity effects [9]. We therefore require an underlying gauge symmetry of the form

$$\mathrm{SM} \times G \, .$$

Whenever a new gauge symmetry is added to the SM gauge group, it is necessary to verify that such an extension is anomaly free. With the above symmetry structure, three new types of anomalies arise:

$$\mathrm{SM}-\mathrm{SM}-G\,, \qquad \mathrm{SM}-G-G\,, \qquad G-G-G\,.$$

The requirement of anomaly freedom at the level of the continuous gauge symmetry G results in the so-called *discrete anomaly conditions* after its breaking to the discrete symmetry $\mathcal{G} \subset G$.

Ibáñez and Ross [10, 11] were the first to carry out a systematic study of these discrete anomaly conditions in the case of $G = \mathrm{U}(1)$ breaking down to $\mathcal{G} = \mathcal{Z}_N$. There the potential anomalies are

$$\begin{array}{lll} \mathrm{SU}(3)_C-\mathrm{SU}(3)_C-\mathrm{U}(1)\,, & \mathrm{SU}(2)_W-\mathrm{SU}(2)_W-\mathrm{U}(1)\,, & \mathrm{Gravity}-\mathrm{Gravity}-\mathrm{U}(1)\,, \\ \mathrm{U}(1)_Y-\mathrm{U}(1)_Y-\mathrm{U}(1)\,, & \mathrm{U}(1)_Y-\mathrm{U}(1)-\mathrm{U}(1)\,, & \mathrm{U}(1)-\mathrm{U}(1)-\mathrm{U}(1)\,, \end{array}$$

where $\mathrm{SU}(3)_C$, $\mathrm{SU}(2)_W$, and $\mathrm{U}(1)_Y$ are the Standard Model gauge groups. The investigation of their discrete versions revealed that the anomalies of the first row severely constrain the allowed anomaly-free discrete gauge symmetries. Under the assumptions that the light fermions of the theory, i.e those particles which do not acquire a mass when the $\mathrm{U}(1)$ breaks down to $\mathcal{Z}_N$, are solely the Standard Model particles, only a finite number of non-equivalent $\mathcal{Z}_N$ symmetries is possible [12]. Adding three right-handed neutrinos to the light particle content, thus requiring pure Dirac neutrinos, allows an infinity of anomaly-free discrete gauge symmetries [13].

An analogous systematic study of the discrete anomaly conditions for the case where $\mathcal{G}$ is a non-Abelian finite group is still lacking. It is the purpose of this article to fill this gap and provide useful constraints on flavor models applying a non-Abelian discrete symmetry.

Since there are only three chiral families in Nature, any candidate finite family group $\mathcal{G}$ should have two- or three-dimensional irreducible representations. This limits the possibilities to finite subgroups of SU(3), SU(2), and SO(3) $\approx$ SU(2)/$\mathcal{Z}_2$. Here we restrict ourselves to the groups $\mathcal{PSL}_2(7)$ [14], $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ [15], $\Delta(27)$ [16–20], $\mathcal{S}_4$ [21–23], $\mathcal{A}_4$ [24–36], $\mathcal{D}_5$ [37, 38], and $\mathcal{S}_3$ [39–45]. The underlying family gauge symmetry G_f must be non-Abelian, the natural candidates being

$$G_f \;=\; \mathrm{SU}(3)\,,\;\mathrm{SU}(2)\,,\;\mathrm{SO}(3)\;.$$

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	Family Group	Irreps	Index	Constraints
High-Energy	G_f	$\boldsymbol{\rho}$	$I(\boldsymbol{\rho})$	anomaly conditions
Low-Energy	$\mathcal{G}$	$\mathbf{r_i}$	$\widetilde{I}(\mathbf{r_i})$	discrete anomaly conditions

Table 1: The notation for the continuous and the discrete family groups, their irreps and the corresponding indices.

Then, the potential anomalies are[1]

$$G_f - G_f - \mathrm{U}(1)_Y\,, \qquad G_f - G_f - G_f\,,$$

where the cubic anomaly is absent for $G_f = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In order to formulate the discrete anomaly conditions, we first need to understand how the irreps $\boldsymbol{\rho}$ of G_f decompose into irreps $\mathbf{r_i}$ of the finite subgroup $\mathcal{G}$. As this decomposition depends on the finite group and its underlying gauge group, one has to discuss each case separately. Furthermore, since the quadratic and the cubic indices of the irreps of G_f enter the original anomaly conditions, it is necessary to introduce the concept of *discrete indices* for irreps of $\mathcal{G}$. Table 1 summarizes our conventions and notation before and after the breaking of G_f into $\mathcal{G}$.

In section 2, we present the general method of consistently defining discrete indices for irreps of $\mathcal{G}$ assuming that the underlying family symmetry is SU(3). We discuss the symmetries $\mathcal{PSL}_2(7)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ explicitly in section 3. The corresponding results for $\mathcal{G} = \Delta(27), \mathcal{S}_4, \mathcal{A}_4, \mathcal{D}_5, \mathcal{S}_3$ are shown in section 4. In section 5 we consider the possibility that $\mathcal{G}$ originates from SO(3), leading to exactly the same discrete quadratic indices as for SU(3). Having defined the discrete indices, we determine the discrete anomaly conditions in section 6. We apply our discrete anomaly conditions to existing examples of flavor models in section 7 and conclude in section 8. Appendices A and B supplement the proof of defining discrete indices consistently.

Finally, for the sake of quick reference for model builders, we summarize our discrete anomaly conditions together with the relevant discrete indices in appendix C.

2. The indices of finite subgroups of SU(3)

It is the purpose of this section to provide a definition of indices for finite subgroups $\mathcal{G}$ of SU(3). The underlying idea is that the finite group should have a gauge origin. As the continuous gauge symmetry SU(3) is broken, its representations $\boldsymbol{\rho}$ decompose into a sum of representations $\mathbf{r_i}$ of $\mathcal{G}$. We will show that one can consistently introduce *discrete indices* for the irreps $\mathbf{r_i}$ of $\mathcal{G}$. These discrete indices can be understood as the vestige of the SU(3) gauge theory which is supposed to be anomaly free. Therefore, they allow an

[1] Upon completion of this work we became aware of ref. [46] which focuses on discrete anomalies of the type $\mathcal{G} - \mathrm{SM} - \mathrm{SM}$, see also [47]. Notice that such an anomaly does not exist in our approach because there is no $G_f - \mathrm{SM} - \mathrm{SM}$ anomaly to begin with.

extension of the well-known discrete anomaly conditions for Abelian symmetries [10 – 13] to the non-Abelian case.

As a starting point, let us briefly recapitulate how the indices of SU(3) irreps are defined. The algebra of SU(3) is given by the Gell-Mann matrices λ_a, $a = 1, \ldots, 8$. The 3×3 matrices satisfy [48, 49]

$$\lambda_a \, \lambda_b \;=\; \frac{2\delta_{ab}}{3} \, \mathbb{1} \;+\; i \, f_{abc} \, \lambda_c \;+\; d_{abc} \, \lambda_c \,,$$

where δ_{ab} denotes the Kronecker symbol, f_{abc} are the (antisymmetric) structure constants and d_{abc} the (symmetric) d-coefficients of SU(3). Denoting the generators of the representation $\boldsymbol{\rho}$ of SU(3) by $T_a^{[\boldsymbol{\rho}]}$, we can determine the following traces

$$\mathrm{Trace}\left(\left\{ T_a^{[\boldsymbol{\rho}]} \,, T_b^{[\boldsymbol{\rho}]} \right\}\right) = \ell(\boldsymbol{\rho}) \, \delta_{ab} \,, \tag{2.1}$$

$$\mathrm{Trace}\left(\left\{ T_a^{[\boldsymbol{\rho}]} \,, T_b^{[\boldsymbol{\rho}]} \right\} T_c^{[\boldsymbol{\rho}]}\right) = A(\boldsymbol{\rho}) \, \frac{d_{abc}}{2} \,, \tag{2.2}$$

with $\{\,,\,\}$ being the anticommutator. $\ell(\boldsymbol{\rho})$ and $A(\boldsymbol{\rho})$ are respectively the quadratic and the cubic index of $\boldsymbol{\rho}$. These two indices correspond to the two fundamental Casimir operators of SU(3). Applying the normalization in which the generators of the irrep $\boldsymbol{\rho} = \mathbf{3}$ are given by $T_a^{[\mathbf{3}]} = \lambda_a/2$ we obtain

$$\ell(\mathbf{3}) \;=\; \ell(\overline{\mathbf{3}}) \;=\; 1 \,, \qquad A(\mathbf{3}) \;=\; -\, A(\overline{\mathbf{3}}) \;=\; 1 \,,$$

for the fundamental irrep $\mathbf{3}$ and its complex conjugate $\overline{\mathbf{3}}$. For higher irreps $\boldsymbol{\rho}$ of SU(3), the indices $I(\boldsymbol{\rho}) = \ell(\boldsymbol{\rho}), A(\boldsymbol{\rho})$ can be calculated recursively from the composition relation [50 – 52]

$$I(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) \;=\; d(\boldsymbol{\rho}) \, I(\boldsymbol{\sigma}) \;+\; I(\boldsymbol{\rho}) \, d(\boldsymbol{\sigma}) \,, \tag{2.3}$$

where $d(\boldsymbol{\rho})$ is the dimension of $\boldsymbol{\rho}$. For the irreps up to dimension 27 one finds the values listed in table 2.

When SU(3) breaks down to the finite subgroup $\mathcal{G}$, the irreps $\boldsymbol{\rho}$ decomposes into irreps $\mathbf{r_i}$ of the finite subgroup with multiplicities a_i. We have

$$\boldsymbol{\rho} \;=\; \bigoplus_i \, a_i \, \mathbf{r_i} \,, \tag{2.4}$$

where the sum is over all irreps of the finite group. Since this breaking process must be consistent with the Kronecker products of $\mathcal{G}$, the irreps $\mathbf{r_i}$ inherit *discrete indices* $\widetilde{I}(\mathbf{r_i}) = \widetilde{\ell}(\mathbf{r_i}), \widetilde{A}(\mathbf{r_i})$ from their parent irreps. Assuming that these discrete indices $\widetilde{I}(\mathbf{r_i})$ are well-defined, we introduce the quantity

$$\mathfrak{I}(\boldsymbol{\rho}) \;=\; a_i \, \widetilde{I}(\mathbf{r_i}) \,, \tag{2.5}$$

to show that

$$I(\boldsymbol{\rho}) \;=\; \mathfrak{I}(\boldsymbol{\rho}) \bmod N_I \,, \tag{2.6}$$

holds true for all irreps $\boldsymbol{\rho}$. The integer N_I depends only on the type of index (quadratic or cubic) and the finite subgroup $\mathcal{G}$. Before proving eq. (2.6) for individual finite groups $\mathcal{G}$, we outline the general procedure.

Irreps $\boldsymbol{\rho}$ of SU(3)	$\ell(\boldsymbol{\rho})$	$A(\boldsymbol{\rho})$
$(10): \quad \mathbf{3}$	1	1
$(01): \quad \overline{\mathbf{3}}$	1	-1
$(20): \quad \mathbf{6}$	5	7
$(02): \quad \overline{\mathbf{6}}$	5	-7
$(11): \quad \mathbf{8}$	6	0
$(30): \quad \mathbf{10}$	15	27
$(21): \quad \mathbf{15}$	20	14
$(40): \quad \mathbf{15'}$	35	77
$(05): \quad \mathbf{21}$	70	-182
$(13): \quad \mathbf{24}$	50	-64
$(22): \quad \mathbf{27}$	54	0

Table 2: The quadratic and the cubic indices of the smallest irreps of SU(3).

Evaluation of eq. (2.6) for the smallest SU(3) irreps can be used to "guess" the discrete indices $\widetilde{I}(\mathbf{r_i})$ and the value of N_I. Once these numbers are given, it is possible to prove by induction that eq. (2.6) is valid for all higher irreps of SU(3) as well.[2] Since these higher irreps can be obtained by successive multiplication with smaller irreps, the inductive step consists in showing the validity of eq. (2.6) for the product $\boldsymbol{\rho} \otimes \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is an SU(3) irrep which decomposes as

$$\boldsymbol{\sigma} \;=\; \bigoplus_i b_i \, \mathbf{r_i} \; . \tag{2.7}$$

It is argued in appendix A that we need only consider $\boldsymbol{\sigma} = \mathbf{3}, \overline{\mathbf{3}}$ to prove our proposition. At this stage, however, we keep our presentation general.

- For $\boldsymbol{\rho} \otimes \boldsymbol{\sigma}$, the left-hand side of eq. (2.6) is obtained from eq. (2.3), yielding

$$\begin{aligned} I(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) &= a_i \, d(\mathbf{r_i}) \, I(\boldsymbol{\sigma}) \;+\; \big(\mathfrak{I}(\boldsymbol{\rho}) \bmod N_I\big) \, d(\boldsymbol{\sigma}) \\ &= a_i \underbrace{\big[d(\mathbf{r_i}) \, I(\boldsymbol{\sigma}) \;+\; \widetilde{I}(\mathbf{r_i}) \, d(\boldsymbol{\sigma})\big]}_{\equiv f_I^i(\boldsymbol{\sigma})} \mod \; d(\boldsymbol{\sigma}) N_I \, , \end{aligned} \tag{2.8}$$

  with $d(\mathbf{r_i})$ denoting the dimension of the irrep $\mathbf{r_i}$. Notice that we have assumed eq. (2.6) for the irrep $\boldsymbol{\rho}$ in the first step. Variation of $\boldsymbol{\rho}$ in this equation changes the parameters a_i whereas the factors $f_I^i(\boldsymbol{\sigma})$ remain unaffected.

- Next we consider the right-hand side of eq. (2.6) for $\boldsymbol{\rho} \otimes \boldsymbol{\sigma}$. This representation decomposes into the irreps of the finite group as

$$\boldsymbol{\rho} \otimes \boldsymbol{\sigma} \;=\; a_i \, b_j \; \mathbf{r_i} \otimes \mathbf{r_j} \;=\; a_i \, b_j \, K_k^{ij} \; \mathbf{r_k} \; . \tag{2.9}$$

  K_k^{ij} are the multiplicities of the irrep $\mathbf{r_k}$ in the Kronecker product $\mathbf{r_i} \otimes \mathbf{r_j}$. We get

$$\mathfrak{I}(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) \;=\; a_i \, b_j \, K_k^{ij} \, \widetilde{I}(\mathbf{r_k}) \;=\; a_i \underbrace{\big[\widetilde{I}(\mathbf{r_i} \otimes b_j \mathbf{r_j})\big]}_{\equiv \mathfrak{f}_I^i(\boldsymbol{\sigma})} \, , \tag{2.10}$$

[2] We thank Dr. Yuji Tachikawa for his inductive proof for the discrete cubic index of $\mathcal{PSL}_2(7)$.

where, in the last step, we require $\widetilde{I}(\cdots)$ to be linear in its argument. Again the factors $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ depend only on $\boldsymbol{\sigma}$ and i but not on $\boldsymbol{\rho}$.

With the above remarks, the proof of eq. (2.6) boils down to showing that

$$f_I^i(\boldsymbol{\sigma}) \;=\; \mathfrak{f}_I^i(\boldsymbol{\sigma}) \bmod N_I \;. \tag{2.11}$$

In the following sections we will discuss various finite subgroups of SU(3), presenting the decomposition of the smallest SU(3) irreps, listing the "guessed" values for N_I and the discrete indices, and finally proving that these definitions satisfy eq. (2.11) for $\boldsymbol{\sigma} = \mathbf{3}, \overline{\mathbf{3}}$. Thus the concept of discrete indices is shown to be consistent.

3. Indices of $\mathcal{PSL}_2(7)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$

3.1 The group $\mathcal{PSL}_2(7)$

As our first example, we discuss the case of $\mathcal{PSL}_2(7)$ which is the unique simple subgroup of SU(3) with complex three-dimensional irreps. Including the singlet, it has six irreps $\mathbf{r_i}$

$$\mathbf{r_0} = \mathbf{1}, \quad \mathbf{r_1} = \mathbf{3}, \quad \mathbf{r_2} = \overline{\mathbf{3}}, \quad \mathbf{r_3} = \mathbf{6}, \quad \mathbf{r_4} = \mathbf{7}, \quad \mathbf{r_5} = \mathbf{8}\;.$$

The decomposition of the SU(3) irreps into these has been worked out in ref. [14]. For the smallest irreps we have:

$SU(3) \supset \mathcal{PSL}_2(7)$
$(10):\ \mathbf{3} = \mathbf{3}$
$(01):\ \overline{\mathbf{3}} = \overline{\mathbf{3}}$
$(20):\ \mathbf{6} = \mathbf{6}$
$(02):\ \overline{\mathbf{6}} = \mathbf{6}$
$(11):\ \mathbf{8} = \mathbf{8}$
$(30):\ \mathbf{10} = \overline{\mathbf{3}} + \mathbf{7}$
$(21):\ \mathbf{15} = \mathbf{7} + \mathbf{8}$
$(40):\ \mathbf{15'} = \mathbf{1} + \mathbf{6} + \mathbf{8}$
$(05):\ \mathbf{21} = \mathbf{3} + \overline{\mathbf{3}} + \mathbf{7} + \mathbf{8}$
$(13):\ \mathbf{24} = \overline{\mathbf{3}} + \mathbf{6} + \mathbf{7} + \mathbf{8}$
$(22):\ \mathbf{27} = \mathbf{6} + \mathbf{6} + \mathbf{7} + \mathbf{8}$

Due to this decomposition, the discrete indices of most $\mathcal{PSL}_2(7)$ irreps $\mathbf{r_i}$ can be simply set equal to the indices of the corresponding SU(3) irreps. Since both, the $\mathbf{6}$ and the $\overline{\mathbf{6}}$ of SU(3) decompose into the $\mathbf{6}$ of $\mathcal{PSL}_2(7)$, we already see that the cubic index $\widetilde{A}(\mathbf{r_i})$ can only be defined modulo $N_A = 14$. As for the $\mathbf{7}$, we observe that eq. (2.6) requires

$$I(\mathbf{10}) \;=\; \widetilde{I}(\overline{\mathbf{3}}) + \widetilde{I}(\mathbf{7}) \bmod N_I\,, \tag{3.1}$$

thus fixing $\widetilde{I}(\mathbf{7})$ modulo N_I. Having defined the values for all $\widetilde{I}(\mathbf{r_i})$, one can easily determine N_I from the higher irreps of SU(3). For the quadratic index we obtain from the $\mathbf{15'}$ that

$N_\ell = 24$. These integers and the discrete indices of the $\mathcal{PSL}_2(7)$ irreps are listed in table 3(a).

Before proving that our assignments are consistent with eq. (2.6) *for all* irreps $\boldsymbol{\rho}$ of SU(3), we consider three examples.

(i) $\boldsymbol{\rho} = \mathbf{3}$. This is a trivial case, since the $\mathbf{3}$ of SU(3) corresponds to the $\mathbf{3}$ of $\mathcal{PSL}_2(7)$. eq. (2.6) then reads

$$\begin{aligned} 1 &= \ell(\mathbf{3}) = \widetilde{\ell}(\mathbf{3}) \bmod 24 = 1 \, , \\ 1 &= A(\mathbf{3}) = \widetilde{A}(\mathbf{3}) \bmod 14 = 1 \, . \end{aligned}$$

(ii) $\boldsymbol{\rho} = \overline{\mathbf{3}}$. Similar to (i) we get

$$\begin{aligned} 1 &= \ell(\overline{\mathbf{3}}) = \widetilde{\ell}(\overline{\mathbf{3}}) \bmod 24 = 1 \, , \\ -1 &= A(\overline{\mathbf{3}}) = \widetilde{A}(\overline{\mathbf{3}}) \bmod 14 = -1 \, . \end{aligned}$$

(iii) $\boldsymbol{\rho} = \mathbf{27}$. This representation of SU(3) decomposes into $\mathbf{6}$+$\mathbf{6}$+$\mathbf{7}$+$\mathbf{8}$ of $\mathcal{PSL}_2(7)$. Inserting the discrete indices of table 3(a) into eq. (2.6) we obtain

$$\begin{aligned} 54 &= \ell(\mathbf{27}) = 2\,\widetilde{\ell}(\mathbf{6}) + \widetilde{\ell}(\mathbf{7}) + \widetilde{\ell}(\mathbf{8}) \bmod 24 = 30 \bmod 24 \, , \\ 0 &= A(\mathbf{27}) = 2\,\widetilde{A}(\mathbf{6}) + \widetilde{A}(\mathbf{7}) + \widetilde{A}(\mathbf{8}) \bmod 14 = 14 \bmod 14 \, . \end{aligned}$$

The first two examples serve as the basis[3] of our proof of eq. (2.6) for the group $\mathcal{PSL}_2(7)$. As discussed in section 2 and appendix A, the inductive step consists in showing that eq. (2.11) holds true for $\boldsymbol{\sigma} = \mathbf{3}, \overline{\mathbf{3}}$:

- The left-hand side, i.e. the factors $f_I^i(\boldsymbol{\sigma})$, can be calculated easily using only the information in table 3(a). We have

$$f_I^i(\mathbf{3}) = d(\mathbf{r_i}) + 3\,\widetilde{I}(\mathbf{r_i}) \, , \tag{3.2}$$

$$f_I^i(\overline{\mathbf{3}}) = (-1)^\kappa\, d(\mathbf{r_i}) + 3\,\widetilde{I}(\mathbf{r_i}) \, , \tag{3.3}$$

  with $\kappa = 0$ (or 2) for the quadratic index $I = \ell$, whereas $\kappa = 1$ (or 3) for the cubic index $I = A$. The explicit values for both types of indices and all six irreps $\mathbf{r_i}$ of $\mathcal{PSL}_2(7)$ are given in the first table of appendix B.

- In order to calculate the right-hand side, i.e. the factors

$$\mathfrak{f}_I^i(\boldsymbol{\sigma}) \;=\; \widetilde{I}(\mathbf{r_i} \otimes b_j \mathbf{r_j}) \, , \tag{3.4}$$

  we need to know the Kronecker products of the finite group. For $\mathcal{PSL}_2(7)$ they can be found in ref. [14]. Since $\boldsymbol{\sigma}$ is constrained to be either $\mathbf{3}$ or $\overline{\mathbf{3}}$, only the following subset of all Kronecker products is necessary.

[3] See appendix A, in particular the Young tableaux of eq. (A.1) with $k = 1$.

Relevant $\mathcal{PSL}_2(7)$ Kronecker Products
$\mathbf{3}\otimes\mathbf{3} = \overline{\mathbf{3}}_a + \mathbf{6}_s$
$\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1} + \mathbf{8}$
$\overline{\mathbf{3}}\otimes\overline{\mathbf{3}} = \mathbf{3}_a + \mathbf{6}_s$
$\mathbf{3}\otimes\mathbf{6} = \overline{\mathbf{3}} + \mathbf{7} + \mathbf{8}$
$\overline{\mathbf{3}}\otimes\mathbf{6} = \mathbf{3} + \mathbf{7} + \mathbf{8}$
$\mathbf{3}\otimes\mathbf{7} = \mathbf{6} + \mathbf{7} + \mathbf{8}$
$\overline{\mathbf{3}}\otimes\mathbf{7} = \mathbf{6} + \mathbf{7} + \mathbf{8}$
$\mathbf{3}\otimes\mathbf{8} = \mathbf{3} + \mathbf{6} + \mathbf{7} + \mathbf{8}$
$\overline{\mathbf{3}}\otimes\mathbf{8} = \overline{\mathbf{3}} + \mathbf{6} + \mathbf{7} + \mathbf{8}$

Thus the factors $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ can be readily determined from

$$\begin{aligned}
a_i\,\mathfrak{f}_I^i(\mathbf{3}) &= a_i\,\widetilde{I}(\mathbf{r_i}\otimes\mathbf{3})\\
&= a_0\,\widetilde{I}(\mathbf{3}) \;+\; a_1\left[\widetilde{I}(\overline{\mathbf{3}})+\widetilde{I}(\mathbf{6})\right] + \;a_2\left[\widetilde{I}(\mathbf{1})+\widetilde{I}(\mathbf{8})\right] + \;a_3\left[\widetilde{I}(\overline{\mathbf{3}})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right]\\
&\quad + a_4\left[\widetilde{I}(\mathbf{6})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right] + \;a_5\left[\widetilde{I}(\mathbf{3})+\widetilde{I}(\mathbf{6})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right]\,, \qquad (3.5)\\
a_i\,\mathfrak{f}_I^i(\overline{\mathbf{3}}) &= a_i\,\widetilde{I}(\mathbf{r_i}\otimes\overline{\mathbf{3}})\\
&= a_0\,\widetilde{I}(\overline{\mathbf{3}}) \;+\; a_1\left[\widetilde{I}(\mathbf{1})+\widetilde{I}(\mathbf{8})\right] + \;a_2\left[\widetilde{I}(\mathbf{3})+\widetilde{I}(\mathbf{6})\right] + \;a_3\left[\widetilde{I}(\mathbf{3})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right]\\
&\quad + a_4\left[\widetilde{I}(\mathbf{6})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right] + \;a_5\left[\widetilde{I}(\overline{\mathbf{3}})+\widetilde{I}(\mathbf{6})+\widetilde{I}(\mathbf{7})+\widetilde{I}(\mathbf{8})\right]\,, \qquad (3.6)
\end{aligned}$$

for both types of indices $I = \ell, A$. Their values are calculated and tabulated in appendix B.

Having obtained the factors $f_I^i(\boldsymbol{\sigma})$ and $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ numerically, we can compare them one by one. Bearing in mind that our calculations are only modulo N_I, we find that eq. (2.11) is truly valid for both the quadratic as well as the cubic index. Furthermore, the comparison also reveals that our values for N_I are the maximally allowed ones. Of course, all statements in this section would remain true if one were to replace all N_I by $N_I' = N_I/p$ where p is an integer. For instance, the cubic index could be defined modulo 7 instead of modulo 14. This completes our proof of eq. (2.6) for the group $\mathcal{PSL}_2(7)$, with the discrete indices given in table 3(a).

3.2 The group $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$

$\mathcal{PSL}_2(7)$ has two maximal subgroups, one of which is the Frobenius group $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, see e.g. ref. [15]. It has the following five irreps $\mathbf{r_i}$.

$$\mathbf{r_0} = \mathbf{1}, \quad \mathbf{r_1} = \mathbf{1}', \quad \mathbf{r_2} = \overline{\mathbf{1}'}, \quad \mathbf{r_3} = \mathbf{3}, \quad \mathbf{r_4} = \overline{\mathbf{3}}\;.$$

The decomposition of SU(3) irreps into these can be easily obtained from the embedding sequence SU(3) $\supset \mathcal{PSL}_2(7) \supset \mathcal{Z}_7 \rtimes \mathcal{Z}_3$ [14], yielding the result:

$\boldsymbol{SU(3)} \supset \boldsymbol{\mathcal{Z}_7 \rtimes \mathcal{Z}_3}$
$(10): \ \mathbf{3} = \mathbf{3}$
$(01): \ \overline{\mathbf{3}} = \overline{\mathbf{3}}$
$(20): \ \mathbf{6} = \mathbf{3}+\overline{\mathbf{3}}$
$(02): \ \overline{\mathbf{6}} = \mathbf{3}+\overline{\mathbf{3}}$
$(11): \ \mathbf{8} = \mathbf{1'}+\overline{\mathbf{1'}}+\mathbf{3}+\overline{\mathbf{3}}$
$(30): \ \mathbf{10} = \mathbf{1}+\mathbf{3}+2\cdot\overline{\mathbf{3}}$
$(21): \ \mathbf{15} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+2\cdot(\mathbf{3}+\overline{\mathbf{3}})$
$(40): \ \mathbf{15'} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+2\cdot(\mathbf{3}+\overline{\mathbf{3}})$
$(05): \ \mathbf{21} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+3\cdot(\mathbf{3}+\overline{\mathbf{3}})$
$(13): \ \mathbf{24} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+3\cdot\mathbf{3}+4\cdot\overline{\mathbf{3}}$
$(22): \ \mathbf{27} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+4\cdot(\mathbf{3}+\overline{\mathbf{3}})$

Notice that $\mathbf{1'}$ and $\overline{\mathbf{1'}}$ always come in pairs in the decomposition of the smallest SU(3) irreps. It is easy to prove this peculiarity for arbitrary irreps of SU(3) by induction. Assume that $\boldsymbol{\rho}$ decomposes as

$$\boldsymbol{\rho} \;=\; a_0\,\mathbf{1} \;+\; a_1\,(\mathbf{1'} \;+\; \overline{\mathbf{1'}}) \;+\; a_3\,\mathbf{3} \;+\; a_4\,\overline{\mathbf{3}}\;. \tag{3.7}$$

Using eq. (2.9) and the $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ Kronecker products [14]

$\boldsymbol{\mathcal{Z}_7 \rtimes \mathcal{Z}_3}$ **Kronecker Products**
$\mathbf{1'} \otimes \mathbf{1'} = \overline{\mathbf{1'}}$
$\mathbf{1'} \otimes \overline{\mathbf{1'}} = \mathbf{1}$
$\mathbf{3} \otimes \mathbf{1'} = \mathbf{3}$
$\mathbf{3} \otimes \overline{\mathbf{1'}} = \mathbf{3}$
$\mathbf{3} \otimes \mathbf{3} = (\mathbf{3} + \overline{\mathbf{3}})_s + \overline{\mathbf{3}}_a$
$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} + \mathbf{1'} + \overline{\mathbf{1'}} + \mathbf{3} + \overline{\mathbf{3}}$

we obtain the decomposition of the representations $\boldsymbol{\rho}\otimes\mathbf{3}$ and $\boldsymbol{\rho}\otimes\overline{\mathbf{3}}$.

$$\begin{aligned}
\boldsymbol{\rho}\otimes\mathbf{3} &= a_4\,\mathbf{1} \;+\; a_4\,(\mathbf{1'} \;+\; \overline{\mathbf{1'}}) \;+\; (a_0+2a_1+a_3+a_4)\,\mathbf{3} \;+\; (2a_3+a_4)\,\overline{\mathbf{3}}\,,\\
\boldsymbol{\rho}\otimes\overline{\mathbf{3}} &= a_3\,\mathbf{1} \;+\; a_3\,(\mathbf{1'} \;+\; \overline{\mathbf{1'}}) \;+\; (a_3+2a_4)\,\mathbf{3} \;+\; (a_0+2a_1+a_3+a_4)\,\overline{\mathbf{3}}\;.
\end{aligned}$$

Of course, these are the decompositions of *reducible* SU(3) representations, i.e. of sums of SU(3) irreps. As argued in appendix A, such a sum contains *only one new* SU(3) irrep. Assuming that the other known irreps decompose with the $\mathbf{1'}$ and $\overline{\mathbf{1'}}$ appearing in pairs, this is true also for the new SU(3) irrep and therefore for all.

Since eq. (3.7) holds for any irrep $\boldsymbol{\rho}$, the discrete indices cannot be defined uniquely. We take this fact into account by introducing the parameters x and y. For physical applications of the discrete indices, it might be convenient to choose specific values, see section 6. At this point, however, we want to stay as general as possible, thus leaving x and y undetermined. It should also be stressed that there is nothing wrong with having non-integer values. Table 3(b) shows the discrete indices of $\mathcal{Z}_7 \rtimes \mathcal{Z}_3 \subset \mathrm{SU}(3)$. The values of N_I for both types of indices are determined by the decomposition of the $\mathbf{6}$.

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(a) Discrete indices of $\mathcal{PSL}_2(7) \subset \mathrm{SU}(3)$.

$\mathcal{PSL}_2(7)$ irreps	$\widetilde{\ell}(\mathbf{r})$ $(N_\ell = 24)$	$\widetilde{A}(\mathbf{r})$ $(N_A = 14)$
$\mathbf{1}$	0	0
$\mathbf{3}$	1	1
$\overline{\mathbf{3}}$	1	-1
$\mathbf{6}$	5	7
$\mathbf{7}$	14	0
$\mathbf{8}$	6	0

(b) Discrete indices of $\mathcal{Z}_7 \rtimes \mathcal{Z}_3 \subset \mathrm{SU}(3)$.

$\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ irreps	$\widetilde{\ell}(\mathbf{r})$ $(N_\ell = 3)$	$\widetilde{A}(\mathbf{r})$ $(N_A = 7)$
$\mathbf{1}$	0	0
$\mathbf{1}'$	x	y
$\overline{\mathbf{1}'}$	$1-x$	$-y$
$\mathbf{3}$	1	1
$\overline{\mathbf{3}}$	1	-1

Table 3: The definition of the discrete indices of the finite groups $\mathcal{PSL}_2(7)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ originating in the continuous group SU(3). x and y can take arbitrary values.

This assignment trivially satisfies eq. (2.6) for $\boldsymbol{\rho} = \mathbf{3}, \overline{\mathbf{3}}$. In order to prove it for all other SU(3) irreps, we need to compare $f_I^i(\boldsymbol{\sigma})$ and $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ for $\boldsymbol{\sigma} = \mathbf{3}, \overline{\mathbf{3}}$. The former, i.e. $f_I^i(\boldsymbol{\sigma})$, is calculated from eqs. (3.2) and (3.3) using table 3(b). $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ on the other hand is determined from eq. (3.4) with the Kronecker products and the discrete indices of $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$. Their explicit values for both types of indices are listed in appendix B. Note that we only need to compare the sum $f_I^{1+2}(\boldsymbol{\sigma}) = f_I^1(\boldsymbol{\sigma}) + f_I^2(\boldsymbol{\sigma})$ with the sum $\mathfrak{f}_I^{1+2}(\boldsymbol{\sigma}) = \mathfrak{f}_I^1(\boldsymbol{\sigma}) + \mathfrak{f}_I^2(\boldsymbol{\sigma})$ because $\mathbf{1}'$ and $\overline{\mathbf{1}'}$ come in pairs in the decomposition of any SU(3) irrep $\boldsymbol{\rho}$. This comparison shows that our definition of the discrete indices for the group $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, given in table 3(b), satisfies eq. (2.6) and is therefore consistent.

4. Indices of $\boldsymbol{\Delta(27)}$, $\boldsymbol{\mathcal{S}_4}$, $\boldsymbol{\mathcal{A}_4}$, $\boldsymbol{\mathcal{D}_5}$, and $\boldsymbol{\mathcal{S}_3}$

In this section we discuss the discrete indices for other finite subgroups of SU(3). To be self-contained, we also list the embedding of their irreps into those of SU(3), as well as their Kronecker products. However, we refrain from showing explicitly that our definitions of discrete indices are consistent with eq. (2.6). This can be done analogously to the previous sections. Our results for the discrete indices are presented in table 4.

4.1 The group $\Delta(27)$

This group, see e.g. ref. [53], has nine one-dimensional irreps, $\mathbf{1_{r,s}}$ with $r, s = 0, 1, 2$, as well as two three-dimensional ones, $\mathbf{3}$ and $\overline{\mathbf{3}}$. For the one-dimensional irreps we also write

$$\begin{array}{lllll} \mathbf{1_0} = \mathbf{1_{0,0}}\,, & \mathbf{1_1} = \mathbf{1_{0,1}}\,, & \mathbf{1_3} = \mathbf{1_{1,0}}\,, & \mathbf{1_5} = \mathbf{1_{1,1}}\,, & \mathbf{1_7} = \mathbf{1_{1,2}}\,, \\ & \mathbf{1_2} = \overline{\mathbf{1}}_\mathbf{1} = \mathbf{1_{0,2}}\,, & \mathbf{1_4} = \overline{\mathbf{1}}_\mathbf{3} = \mathbf{1_{2,0}}\,, & \mathbf{1_6} = \overline{\mathbf{1}}_\mathbf{5} = \mathbf{1_{2,2}}\,, & \mathbf{1_8} = \overline{\mathbf{1}}_\mathbf{7} = \mathbf{1_{2,1}}\,. \end{array}$$

With this notation, the Kronecker products and the decomposition of the smallest SU(3) irreps are given as:

$\Delta(27)$ Kronecker Products
$\mathbf{1_{r,s}} \otimes \mathbf{1_{r',s'}} = \mathbf{1_{r+r',s+s'}}$
$\mathbf{3} \otimes \mathbf{1_j} = \mathbf{3}$
$\overline{\mathbf{3}} \otimes \mathbf{1_j} = \overline{\mathbf{3}}$
$\mathbf{3} \otimes \mathbf{3} = 3 \cdot \overline{\mathbf{3}}$
$\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = 3 \cdot \mathbf{3}$
$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1_0} + \sum_{j=1}^{8} \mathbf{1_j}$

$SU(3) \supset \Delta(27)$
$(10): \mathbf{3} = \mathbf{3}$
$(01): \overline{\mathbf{3}} = \overline{\mathbf{3}}$
$(20): \mathbf{6} = 2 \cdot \overline{\mathbf{3}}$
$(02): \overline{\mathbf{6}} = 2 \cdot \mathbf{3}$
$(11): \mathbf{8} = \sum_{j=1}^{8} \mathbf{1_j}$
$(30): \mathbf{10} = 2 \cdot \mathbf{1_0} + \sum_{j=1}^{8} \mathbf{1_j}$
$(21): \mathbf{15} = 5 \cdot \mathbf{3}$
$(40): \mathbf{15'} = 5 \cdot \mathbf{3}$
$(05): \mathbf{21} = 7 \cdot \mathbf{3}$
$(13): \mathbf{24} = 8 \cdot \mathbf{3}$
$(22): \mathbf{27} = 3 \cdot (\mathbf{1_0} + \sum_{j=1}^{8} \mathbf{1_j})$

On the right-hand side of the Kronecker product for the one-dimensional irreps, the sums $r+r'$ and $s+s'$ are modulo 3. Similar to the $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ case, one can easily show that the one-dimensional irreps $\mathbf{1_j}$ with $j = 1, \ldots, 8$ occur always collectively in the decomposition of SU(3) irreps. The resulting ambiguity in the definition of the corresponding discrete indices is expressed by introducing the parameters x_k and y_k with $k = 1, \ldots, 7$ in table 4(a). The decomposition of the **6** fixes the values of N_I.

4.2 The group $\mathcal{S}_4$

Besides $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, this group is the second maximal subgroup of $\mathcal{PSL}_2(7)$. It has five irreps.

$$\mathbf{r_0} = \mathbf{1}, \quad \mathbf{r_1} = \mathbf{1'}, \quad \mathbf{r_2} = \mathbf{2}, \quad \mathbf{r_3} = \mathbf{3_1}, \quad \mathbf{r_4} = \mathbf{3_2} \;.$$

The Kronecker products of $\mathcal{S}_4$ and its embedding into SU(3) are [14, 21]:

$\mathcal{S}_4$ Kronecker Products
$\mathbf{1'} \otimes \mathbf{1'} = \mathbf{1}$
$\mathbf{2} \otimes \mathbf{1'} = \mathbf{2}$
$\mathbf{3_1} \otimes \mathbf{1'} = \mathbf{3_2}$
$\mathbf{3_2} \otimes \mathbf{1'} = \mathbf{3_1}$
$\mathbf{2} \otimes \mathbf{2} = (\mathbf{1} + \mathbf{2})_s + (\mathbf{1'})_a$
$\mathbf{2} \otimes \mathbf{3_i} = \mathbf{3_1} + \mathbf{3_2}$
$\mathbf{3_i} \otimes \mathbf{3_i} = (\mathbf{1} + \mathbf{2} + \mathbf{3_1})_s + (\mathbf{3_2})_a$
$\mathbf{3_1} \otimes \mathbf{3_2} = \mathbf{1'} + \mathbf{2} + \mathbf{3_1} + \mathbf{3_2}$

$SU(3) \supset \mathcal{S}_4$
$(10): \mathbf{3} = \mathbf{3_2}$
$(01): \overline{\mathbf{3}} = \mathbf{3_2}$
$(20): \mathbf{6} = \mathbf{1} + \mathbf{2} + \mathbf{3_1}$
$(02): \overline{\mathbf{6}} = \mathbf{1} + \mathbf{2} + \mathbf{3_1}$
$(11): \mathbf{8} = \mathbf{2} + \mathbf{3_1} + \mathbf{3_2}$
$(30): \mathbf{10} = \mathbf{1'} + \mathbf{3_1} + 2 \cdot \mathbf{3_2}$
$(21): \mathbf{15} = \mathbf{1'} + \mathbf{2} + 2 \cdot (\mathbf{3_1} + \mathbf{3_2})$
$(40): \mathbf{15'} = 2 \cdot (\mathbf{1} + \mathbf{2} + \mathbf{3_1}) + \mathbf{3_2}$
$(05): \mathbf{21} = \mathbf{1'} + \mathbf{2} + 2 \cdot \mathbf{3_1} + 4 \cdot \mathbf{3_2}$
$(13): \mathbf{24} = \mathbf{1} + \mathbf{1'} + 2 \cdot \mathbf{2} + 3 \cdot (\mathbf{3_1} + \mathbf{3_2})$
$(22): \mathbf{27} = 2 \cdot \mathbf{1} + \mathbf{1'} + 3 \cdot \mathbf{2} + 4 \cdot \mathbf{3_1} + 2 \cdot \mathbf{3_2}$

Notice that the occurrence of both $\mathbf{1'}$ and $\mathbf{2}$ is always accompanied by the irrep $\mathbf{3_1}$ in the decomposition of the smallest SU(3) irreps. Again, it is easy to prove this for all irreps of SU(3) by induction. Assume that $\boldsymbol{\rho}$ decomposes as

$$\boldsymbol{\rho} \;=\; a_0\,\mathbf{1} \;+\; a_1\,(\mathbf{1'} + \mathbf{3_1}) \;+\; a_2\,(\mathbf{2} + \mathbf{3_1}) \;+\; a_4\,\mathbf{3_2} \;. \tag{4.1}$$

Since $\mathbf{3}$ and $\overline{\mathbf{3}}$ both correspond to the same $\mathcal{S}_4$ irrep $\mathbf{3_2}$, the two SU(3) representations $\boldsymbol{\rho}\otimes\mathbf{3}$ and $\boldsymbol{\rho}\otimes\overline{\mathbf{3}}$ have the same decomposition. It is obtained from the Kronecker products, yielding

$$a_4\,\mathbf{1} \;+\; (a_1+a_2)\,(\mathbf{1'}+\mathbf{3_1}) \;+\; (a_1+a_2+a_4)\,(\,\mathbf{2}+\mathbf{3_1}) \;+\; (a_0+a_1+2a_2+a_4)\,\mathbf{3_2}\,,$$

which is of the same structure as eq. (4.1). Due to this general property of the embedding of $\mathcal{S}_4$ into SU(3), the discrete indices are not defined uniquely. The values for $\widetilde{I}(\mathbf{2}+\mathbf{3_1})$ and $\widetilde{I}(\mathbf{1'}+\mathbf{3_1})$ are given by the $\mathbf{6}$ and the $\mathbf{10}$ of SU(3), respectively. The $\mathbf{15'}$ then determines N_ℓ to be 24 for the quadratic index, while $N_A=2$ for the cubic index because both the $\mathbf{3}$ and the $\overline{\mathbf{3}}$ decompose as a $\mathbf{3_2}$ of $\mathcal{S}_4$. The results are shown in table 4(b), with the ambiguity in the definitions parameterized by x and y.

4.3 The group $\mathcal{A}_4$

This group is the most popular group in flavor model building. It is a subgroup of $\mathcal{S}_4$ and has four irreps.

$$\mathbf{r_0}=\mathbf{1},\quad \mathbf{r_1}=\mathbf{1'},\quad \mathbf{r_2}=\overline{\mathbf{1'}},\quad \mathbf{r_3}=\mathbf{3}\;.$$

The Kronecker products of $\mathcal{A}_4$ are listed throughout the literature. The decomposition of the smallest SU(3) irreps can be worked out easily from $\mathcal{A}_4$'s embedding in $\mathcal{S}_4$ [14].

$\mathcal{A}_4$ Kronecker Products
$\mathbf{1'}\otimes\mathbf{1'} = \overline{\mathbf{1'}}$
$\mathbf{1'}\otimes\overline{\mathbf{1'}} = \mathbf{1}$
$\mathbf{3}\otimes\mathbf{1'} = \mathbf{3}$
$\mathbf{3}\otimes\mathbf{3} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+2\cdot\mathbf{3}$

$SU(3)\supset\mathcal{A}_4$
$(10):\ \mathbf{3} = \mathbf{3}$
$(01):\ \overline{\mathbf{3}} = \mathbf{3}$
$(20):\ \mathbf{6} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+\mathbf{3}$
$(02):\ \overline{\mathbf{6}} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+\mathbf{3}$
$(11):\ \mathbf{8} = \mathbf{1'}+\overline{\mathbf{1'}}+2\cdot\mathbf{3}$
$(30):\ \mathbf{10} = \mathbf{1}+3\cdot\mathbf{3}$
$(21):\ \mathbf{15} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+4\cdot\mathbf{3}$
$(40):\ \mathbf{15'} = 2\cdot(\mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}})+3\cdot\mathbf{3}$
$(05):\ \mathbf{21} = \mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+6\cdot\mathbf{3}$
$(13):\ \mathbf{24} = 2\cdot(\mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}})+6\cdot\mathbf{3}$
$(22):\ \mathbf{27} = 3\cdot(\mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}})+6\cdot\mathbf{3}$

In this case, the irreps $\mathbf{1'}$ and $\overline{\mathbf{1'}}$ always come in pairs in the decomposition of the SU(3) irreps. The discrete indices, listed in table 4(c), are again not uniquely determined. As before $N_A=2$, while the $\mathbf{10}$ fixes N_ℓ to be 12.

4.4 The group $\mathcal{D}_5$

The dihedral group $\mathcal{D}_5$ has also been used as a family group. Its four irreps are

$$\mathbf{r_0}=\mathbf{1},\quad \mathbf{r_1}=\mathbf{1'},\quad \mathbf{r_2}=\mathbf{2_1},\quad \mathbf{r_3}=\mathbf{2_2}\;.$$

The Kronecker products of $\mathcal{D}_5$ as well as its embedding in SU(3) can be found in refs. [37].

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$\mathcal{D}_5$ **Kronecker Products**
$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}$
$\mathbf{1}' \otimes \mathbf{2_i} = \mathbf{2_i}$
$\mathbf{2_1} \otimes \mathbf{2_1} = \mathbf{1} + \mathbf{1}' + \mathbf{2_2}$
$\mathbf{2_2} \otimes \mathbf{2_2} = \mathbf{1} + \mathbf{1}' + \mathbf{2_1}$
$\mathbf{2_1} \otimes \mathbf{2_2} = \mathbf{2_1} + \mathbf{2_2}$

$\boldsymbol{SU(3) \supset \mathcal{D}_5}$
$(10): \ \mathbf{3} = \mathbf{1}' + \mathbf{2_1}$
$(01): \ \overline{\mathbf{3}} = \mathbf{1}' + \mathbf{2_1}$
$(20): \ \mathbf{6} = 2 \cdot \mathbf{1} + \mathbf{2_1} + \mathbf{2_2}$
$(02): \ \overline{\mathbf{6}} = 2 \cdot \mathbf{1} + \mathbf{2_1} + \mathbf{2_2}$
$(11): \ \mathbf{8} = \mathbf{1} + \mathbf{1}' + 2 \cdot \mathbf{2_1} + \mathbf{2_2}$
$(30): \ \mathbf{10} = 2 \cdot (\mathbf{1}' + \mathbf{2_1} + \mathbf{2_2})$
$(21): \ \mathbf{15} = \mathbf{1} + 2 \cdot \mathbf{1}' + 3 \cdot (\mathbf{2_1} + \mathbf{2_2})$
$(40): \ \mathbf{15}' = 3 \cdot (\mathbf{1} + \mathbf{2_1} + \mathbf{2_2})$
$(05): \ \mathbf{21} = \mathbf{1} + 4 \cdot (\mathbf{1}' + \mathbf{2_1} + \mathbf{2_2})$
$(13): \ \mathbf{24} = 2 \cdot (\mathbf{1} + \mathbf{1}') + 5 \cdot (\mathbf{2_1} + \mathbf{2_2})$
$(22): \ \mathbf{27} = 4 \cdot \mathbf{1} + \mathbf{1}' + 5 \cdot \mathbf{2_1} + 6 \cdot \mathbf{2_2}$

Another alternative embedding of $\mathcal{D}_5$ in SU(3) is obtained by exchanging the representations $\mathbf{2_1} \leftrightarrow \mathbf{2_2}$. However, we only spell out the results for the choice shown above. First notice that the smallest irreps $\boldsymbol{\rho}$ of SU(3) all decompose as

$$\boldsymbol{\rho} \ = \ a_0 \, \mathbf{1} \ + \ a_1 \, \mathbf{1}' \ + \ a_2 \, \mathbf{2_1} \ + \ a_3 \, \mathbf{2_2} \, ,$$

with $a_1 + a_2 + a_3$ being even. This can be verified for higher irreps by examining the decomposition of $\boldsymbol{\rho} \otimes \mathbf{3}$ and $\boldsymbol{\rho} \otimes \overline{\mathbf{3}}$, for both of which we find

$$(a_1 + a_2) \, \mathbf{1} \ + \ \underbrace{(a_0 + a_2)}_{a_1'} \, \mathbf{1}' \ + \ \underbrace{(a_0 + a_1 + a_2 + a_3)}_{a_2'} \, \mathbf{2_1} \ + \ \underbrace{(a_2 + 2a_3)}_{a_3'} \, \mathbf{2_2} \ .$$

Obviously, the sum $a_1' + a_2' + a_3'$ is even again. This fact has to be taken into account in the inductive proof of eq. (2.6) where $f_I^i(\boldsymbol{\sigma})$ and $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ are calculated and compared in a table similar to those shown in appendix B.

The assignment of the discrete indices is somewhat more involved because the $\mathbf{3}$ of SU(3) decomposes into two irreps of $\mathcal{D}_5$. Writing

$$\widetilde{I}(\mathbf{1}') = \alpha \, , \qquad \widetilde{I}(\mathbf{2_1}) = \beta \, , \qquad \widetilde{I}(\mathbf{2_2}) = \gamma \, ,$$

we get

$$\begin{aligned} I(\mathbf{3}) \ = \ I(\overline{\mathbf{3}}) &= \alpha \ + \ \beta \mod N_I \, , \\ I(\mathbf{6}) &= \beta \ + \ \gamma \mod N_I \, , \\ I(\mathbf{15}') &= 3 \, (\beta \ + \ \gamma) \mod N_I \, , \\ I(\mathbf{27}) &= \alpha \ + \ 5 \beta \ + \ 6 \gamma \mod N_I \ . \end{aligned}$$

For the quadratic index, N_ℓ is determined by comparing the $\mathbf{15}'$ with three copies of the $\mathbf{6}$, yielding $N_\ell = 20$. For the cubic index we have $N_A = 2$. The value of α can be calculated from

$$I(\mathbf{3}) \ + \ I(\mathbf{27}) \ - \ 6 \, I(\mathbf{6}) \ = \ 2 \, \alpha \mod N_I \ .$$

Due to the mod N_I we get two discrete solutions, parameterized by $\xi = 0, 1$ and $\zeta = 0, 1$, respectively. The values for β and γ are then easily determined from the $\mathbf{3}$ and the $\mathbf{6}$. Table 4(d) lists the results. Notice that these indices are *not* integer.

4.5 The group $\mathcal{S}_3$

Finally, the last group we consider is $\mathcal{S}_3$ with three irreps

$$\mathbf{r_0} = \mathbf{1}, \quad \mathbf{r_1} = \mathbf{1}', \quad \mathbf{r_2} = \mathbf{2} \ .$$

The Kronecker products of $\mathcal{S}_3$ and its embedding in SU(3) is given below.

$\mathcal{S}_3$ **Kronecker Products**
$\mathbf{1}' \otimes \mathbf{1}' \ = \ \mathbf{1}$
$\mathbf{1}' \otimes \mathbf{2} \ = \ \mathbf{2}$
$\mathbf{2} \otimes \mathbf{2} \ = \ \mathbf{1} + \mathbf{1}' + \mathbf{2}$

$\boldsymbol{SU(3)} \ \supset \ \boldsymbol{\mathcal{S}_3}$
$(10): \ \mathbf{3} \ = \ \mathbf{1}' + \mathbf{2}$
$(01): \ \overline{\mathbf{3}} \ = \ \mathbf{1}' + \mathbf{2}$
$(20): \ \mathbf{6} \ = \ 2 \cdot (\mathbf{1} + \mathbf{2})$
$(02): \ \overline{\mathbf{6}} \ = \ 2 \cdot (\mathbf{1} + \mathbf{2})$
$(11): \ \mathbf{8} \ = \ \mathbf{1} + \mathbf{1}' + 3 \cdot \mathbf{2}$
$(30): \ \mathbf{10} \ = \ \mathbf{1} + 3 \cdot (\mathbf{1}' + \mathbf{2})$
$(21): \ \mathbf{15} \ = \ 2 \cdot \mathbf{1} + 3 \cdot \mathbf{1}' + 5 \cdot \mathbf{2}$
$(40): \ \mathbf{15}' \ = \ 4 \cdot \mathbf{1} + \mathbf{1}' + 5 \cdot \mathbf{2}$
$(05): \ \mathbf{21} \ = \ 2 \cdot \mathbf{1} + 5 \cdot \mathbf{1}' + 7 \cdot \mathbf{2}$
$(13): \ \mathbf{24} \ = \ 4 \cdot (\mathbf{1} + \mathbf{1}') + 8 \cdot \mathbf{2}$
$(22): \ \mathbf{27} \ = \ 6 \cdot \mathbf{1} + 3 \cdot \mathbf{1}' + 9 \cdot \mathbf{2}$

Analogous to the group $\mathcal{D}_5$, the SU(3) irreps decompose into irreps of $\mathcal{S}_3$ such that the sum of the multiplicities of $\mathbf{1}'$ and $\mathbf{2}$ is even.

For the quadratic index, N_ℓ is obtained by comparing the **10** with three copies of the **3**, yielding $N_\ell = 12$. For the cubic index we have $N_A = 2$. The **6** determines $\widetilde{I}(\mathbf{2})$, again with two discrete solutions parameterized by ξ and ζ. Then, $\widetilde{I}(\mathbf{1}')$ can be calculated from the **3**. The resulting discrete indices of $\mathcal{S}_3$ are given in table 4(e).

5. $\boldsymbol{\mathcal{S}_4}$, $\boldsymbol{\mathcal{A}_4}$, $\boldsymbol{\mathcal{D}_5}$, and $\boldsymbol{\mathcal{S}_3}$ as subgroups of SO(3)

So far, we have considered $\mathcal{G}$ to be the remnant of a high-energy SU(3) family symmetry. In fact, this is the only possibility for the finite groups $\mathcal{PSL}_2(7)$, $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, and $\Delta(27)$. On the other hand, the groups $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ can alternatively be embedded into SO(3).[4] Since SO(3) = SU(2)/$\mathcal{Z}_2$, the indices of the SO(3) irreps are proportional to the indices of the odd-dimensional irreps of SU(2). For SU(2), cubic indices are absent, and the quadratic indices are defined analogously to the SU(3) case

$$\text{Trace}\left(\left\{ T_a^{[\boldsymbol{\rho}]} , T_b^{[\boldsymbol{\rho}]} \right\}\right) \ = \ \ell(\boldsymbol{\rho})\, \delta_{ab} \ . \tag{5.1}$$

Choosing $T_a^{[\mathbf{2}]} = \sigma_a/2$, with σ_a denoting the Pauli matrices, the quadratic index of the fundamental irrep **2** is normalized to one. The quadratic indices of all higher irreps $\boldsymbol{\rho}$ of SU(2) can then be obtained recursively from

$$\ell(\boldsymbol{\rho} \otimes \mathbf{2}) \ = \ d(\boldsymbol{\rho})\, \ell(\mathbf{2}) \ + \ \ell(\boldsymbol{\rho})\, d(\mathbf{2}) \ = \ d(\boldsymbol{\rho}) \ + \ 2\, \ell(\boldsymbol{\rho}) \ . \tag{5.2}$$

[4] A non-trivial embedding into SU(2) is not possible since the **2** as well as the other even-dimensional irreps of SU(2) are spinor-like, whereas the irreps of $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ are not.

(a) Discrete indices of $\Delta(27) \subset \mathrm{SU}(3)$; $k = 1, \ldots, 7$.

$\Delta(27)$ irreps	$\widetilde{\ell}(\mathbf{r})$ ($N_\ell = 3$)	$\widetilde{A}(\mathbf{r})$ ($N_A = 9$)
$\mathbf{1_0}$	0	0
$\mathbf{1_k}$	x_k	y_k
$\mathbf{1_8}$	$-\sum_{k=1}^{7} x_k$	$-\sum_{k=1}^{7} y_k$
$\mathbf{3}$	1	1
$\overline{\mathbf{3}}$	1	-1

(b) Discrete indices of $\mathcal{S}_4 \subset \mathrm{SU}(3)$.

$\mathcal{S}_4$ irreps	$\widetilde{\ell}(\mathbf{r})$ ($N_\ell = 24$)	$\widetilde{A}(\mathbf{r})$ ($N_A = 2$)
$\mathbf{1}$	0	0
$\mathbf{1}'$	$13 - x$	$1 - y$
$\mathbf{2}$	$5 - x$	$1 - y$
$\mathbf{3_1}$	x	y
$\mathbf{3_2}$	1	1

(c) Discrete indices of $\mathcal{A}_4 \subset \mathrm{SU}(3)$.

$\mathcal{A}_4$ irreps	$\widetilde{\ell}(\mathbf{r})$ ($N_\ell = 12$)	$\widetilde{A}(\mathbf{r})$ ($N_A = 2$)
$\mathbf{1}$	0	0
$\mathbf{1}'$	x	y
$\overline{\mathbf{1}'}$	$4 - x$	$-y$
$\mathbf{3}$	1	1

(d) Discrete indices of $\mathcal{D}_5 \subset \mathrm{SU}(3)$.

$\mathcal{D}_5$ irreps	$\widetilde{\ell}(\mathbf{r})$ ($N_\ell = 20$)	$\widetilde{A}(\mathbf{r})$ ($N_A = 2$)
$\mathbf{1}$	0	0
$\mathbf{1}'$	$(5+\xi N_\ell)/2$	$(1+\zeta N_A)/2$
$\mathbf{2_1}$	$(-3+\xi N_\ell)/2$	$(1+\zeta N_A)/2$
$\mathbf{2_2}$	$(13+\xi N_\ell)/2$	$(1+\zeta N_A)/2$

(e) Discrete indices of $\mathcal{S}_3 \subset \mathrm{SU}(3)$.

$\mathcal{S}_3$ irreps	$\widetilde{\ell}(\mathbf{r})$ ($N_\ell = 12$)	$\widetilde{A}(\mathbf{r})$ ($N_A = 2$)
$\mathbf{1}$	0	0
$\mathbf{1}'$	$(-3+\xi N_\ell)/2$	$(1+\zeta N_A)/2$
$\mathbf{2}$	$(5+\xi N_\ell)/2$	$(1+\zeta N_A)/2$

Table 4: The definition of the discrete quadratic and cubic indices of various finite subgroups of SU(3), namely $\Delta(27)$, $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$. As some irreps of the finite groups do not occur independently from other irreps in the decomposition of SU(3) irreps, the definitions of the discrete indices are not always unique. Where present, this ambiguity is parameterized by x_k, y_k; x, y; $\xi = 0, 1$, and $\zeta = 0, 1$, respectively.

The indices of the odd-dimensional irreps turn out to be multiples of four. Hence, a change of normalization yields the quadratic indices for the smallest irreps of SO(3) shown in table 5.

In order to define the discrete indices for the irreps $\mathbf{r_i}$ of $\mathcal{G} \subset \mathrm{SO}(3)$, we must first determine how the irreps $\boldsymbol{\rho}$ decompose. For the smallest SO(3) irreps, the results are summarized below.

Irreps $\boldsymbol{\rho}$ of SO(3)	$\ell(\boldsymbol{\rho})$
$\mathbf{3}$	1
$\mathbf{5}$	5
$\mathbf{7}$	14
$\mathbf{9}$	30
$\mathbf{11}$	55

Table 5: The quadratic indices of the smallest irreps of SO(3).

$SO(3)$	$\mathcal{S}_4$	$\mathcal{A}_4$	$\mathcal{D}_5$	$\mathcal{S}_3$
$\mathbf{3}$	$\mathbf{3_2}$	$\mathbf{3}$	$\mathbf{1'}+\mathbf{2_1}$	$\mathbf{1'}+\mathbf{2}$
$\mathbf{5}$	$\mathbf{2}+\mathbf{3_1}$	$\mathbf{1'}+\overline{\mathbf{1'}}+\mathbf{3}$	$\mathbf{1}+\mathbf{2_1}+\mathbf{2_2}$	$\mathbf{1}+2\cdot\mathbf{2}$
$\mathbf{7}$	$\mathbf{1'}+\mathbf{3_1}+\mathbf{3_2}$	$\mathbf{1}+2\cdot\mathbf{3}$	$\mathbf{1'}+\mathbf{2_1}+2\cdot\mathbf{2_2}$	$\mathbf{1}+2\cdot(\mathbf{1'}+\mathbf{2})$
$\mathbf{9}$	$\mathbf{1}+\mathbf{2}+\mathbf{3_1}+\mathbf{3_2}$	$\mathbf{1}+\mathbf{1'}+\overline{\mathbf{1'}}+2\cdot\mathbf{3}$	$\mathbf{1}+2\cdot(\mathbf{2_1}+\mathbf{2_2})$	$2\cdot\mathbf{1}+\mathbf{1'}+3\cdot\mathbf{2}$
$\mathbf{11}$	$\mathbf{2}+\mathbf{3_1}+2\cdot\mathbf{3_2}$	$\mathbf{1'}+\overline{\mathbf{1'}}+3\cdot\mathbf{3}$	$\mathbf{1}+2\cdot(\mathbf{1'}+\mathbf{2_1}+\mathbf{2_2})$	$\mathbf{1}+2\cdot\mathbf{1'}+4\cdot\mathbf{2}$

Regarding $\mathcal{D}_5$ there exists an alternative embedding with the irreps $\mathbf{2_1}$ and $\mathbf{2_2}$ interchanged. Similar to the case of SU(3), it is easy to show that the irreps $\boldsymbol{\rho}$ of SO(3) decompose into irreps $\mathbf{r_i}$ of $\mathcal{G}$ with the multiplicities a_i constrained by linear relation. It turns our that these are identical to the relations we obtained when embedding $\mathcal{G}$ into SU(3).

- $\mathcal{S}_4$: The $\mathbf{1'}$ as well as the $\mathbf{2}$ are always accompanied by an irrep $\mathbf{3_1}$.
- $\mathcal{A}_4$: The $\mathbf{1'}$ and the $\overline{\mathbf{1'}}$ always come in pairs.
- $\mathcal{D}_5$: The sum of the multiplicities of $\mathbf{1'}$, $\mathbf{2_1}$, and $\mathbf{2_2}$ is even.
- $\mathcal{S}_3$: The sum of the multiplicities of $\mathbf{1'}$ and $\mathbf{2}$ is even.

Therefore, the discrete indices are not defined uniquely. From the decompositions of the smallest SO(3) irreps we can determine $\widetilde{\ell}(\mathbf{r_i})$, with the arbitrariness parameterized by x and $\xi = 0, 1$, respectively.

- $\mathcal{S}_4$: First we set $\widetilde{\ell}(\mathbf{3_2}) = 1$. Introducing the parameter $x = \widetilde{\ell}(\mathbf{3_1})$, we find from the decomposition of the $\mathbf{5}$ that $\widetilde{\ell}(\mathbf{2}) = 5 - x$. Similarly, the $\mathbf{7}$ fixes $\widetilde{\ell}(\mathbf{1'}) = 13 - x$. Inserting these values for the discrete indices into the decomposition of the $\mathbf{9}$, we see that the quadratic indices can only be defined modulo $N_\ell = 24$.
- $\mathcal{A}_4$: Here we have $\widetilde{\ell}(\mathbf{3}) = 1$. Defining $x = \widetilde{\ell}(\mathbf{1'})$, the decomposition of the $\mathbf{5}$ yields $\widetilde{\ell}(\overline{\mathbf{1'}}) = 4 - x$. With these values, one finds that $N_\ell = 12$.
- $\mathcal{D}_5$: Comparing the decomposition of the $\mathbf{5}$ and the $\mathbf{9}$ shows that $N_\ell = 20$. The value for $2\,\widetilde{\ell}(\mathbf{2_2}) = 13 \bmod N_\ell$ is obtained from combining the SO(3) irreps $\mathbf{3}$ and $\mathbf{7}$. With $\xi = 0, 1$ this gives $\widetilde{\ell}(\mathbf{2_2}) = (13 + \xi N_\ell)/2$. Then the $\mathbf{5}$ fixes $\widetilde{\ell}(\mathbf{2_1}) = (-3 + \xi N_\ell)/2$. Finally, from the $\mathbf{3}$ we get $\widetilde{\ell}(\mathbf{1'}) = (5 + \xi N_\ell)/2$.

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- $\mathcal{S}_3$: Comparison of the SO(3) irreps **3** and **7** yields $N_\ell = 12$. From the **5** we determine $\widetilde{\ell}(\mathbf{2}) = (5 + \xi N_\ell)/2$. Then we find $\widetilde{\ell}(\mathbf{1}') = (-3 + \xi N_\ell)/2$ from the **3**.

Remarkably, all these quadratic indices are identical to the results obtained in section 4 where $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ are considered to be subgroups of SU(3). Concerning the quadratic indices it therefore does not make any difference at all whether $\mathcal{G}$ originates in SU(3) or SO(3). Of course, in the latter case the cubic indices are absent.

6. Discrete anomaly conditions

Having defined the discrete indices for the finite groups $\mathcal{PSL}_2(7)$, $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, $\Delta(27)$, $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$, we are now in a position to formulate the corresponding discrete anomaly conditions. Our starting point is to require anomaly freedom of the underlying continuous family symmetry G_f. Under the assumption that $G_f = \mathrm{SU}(3)$, we obtain two anomaly cancellation conditions

$$\sum_k \ell_k Y_k \;=\; 0\,, \qquad \sum_k A_k \;=\; 0\;.$$

Here k labels the fermions of the complete theory, with ℓ_k and A_k being the quadratic and the cubic index corresponding to the particle's SU(3) irrep. Y_k denotes the hypercharge in the normalization where the left-handed quark doublet has $Y_Q = 1$. In the following, we further assume that no particle k has fractional hypercharge in this normalization.[5] After the breakdown of SU(3) to the non-Abelian finite symmetry $\mathcal{G}$, the SU(3) irreps decompose into irreps of $\mathcal{G}$. Labeling these by i, the discrete anomaly cancellation conditions can be obtained from

$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;+\; \sum_{i=\text{massive}} \widetilde{\ell}_i\, Y_i = 0 \text{ mod } N_\ell\,, \tag{6.1}$$

$$\sum_{i=\text{light}} \widetilde{A}_i \;+\; \sum_{i=\text{massive}} \widetilde{A}_i = 0 \text{ mod } N_A\,, \tag{6.2}$$

with N_I depending on the specific group $\mathcal{G}$. For $G_f = \mathrm{SU}(2)$ or SO(3), the cubic anomaly does not exist so that we are left with eq. (6.1) only. In the following, we evaluate the sums over the massive degrees of freedom in eqs. (6.1) and (6.2), showing that they can be incorporated into the right-hand side, in some cases changing the value of N_I. Thus we are lead to the discrete anomaly conditions which constrain the irreps of $\mathcal{G}$ assigned to the light fermions.

6.1 Mass terms and their effects on the anomaly conditions

Before elaborating on the non-Abelian case, it might be useful to remind ourselves of how massive fermions enter the anomaly equations in a scenario where the discrete symmetry is $\mathcal{Z}_N$ [10, 11]. Such an Abelian discrete symmetry arises when a U(1) gauge symmetry gets

[5]Particles with fractional hypercharge in this normalization are electrically charged. Furthermore, they cannot decay into Standard Model particles alone, so that the lightest such particle would be stable. Since dark matter should be neutral, their existence is disfavored.

spontaneously broken by the vacuum expectation value (VEV) of a SM singlet field with U(1) charge N. As a result of this breaking, some fermions, the so-called massive fermions, will have a bilinear mass term whose U(1) charge is an integer multiple of N. Using the standard conventions, a pair of massive fermionic fields can only contribute an integer multiple of $N/2$ to the anomaly equations. Therefore the discrete anomaly conditions are given modulo N, with N directly related to the spontaneous breaking of the continuous U(1) symmetry.

The situation becomes more involved with non-Abelian symmetries for two reasons. First, the details of how G_f breaks down to $\mathcal{G}$ are ambiguous as can be seen, e.g., from the decomposition of SU(3) irreps into irreps of $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ [cf. above eq. (3.7)]. Even when restricting to the irreps of SU(3) up to **27**, there are six irreps which can acquire a VEV (in a suitable direction) and leave the discrete symmetry $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ unbroken. Instead of being related to the breaking of the continuous to the discrete family symmetry, the values for N_I in the modulo N_I of the discrete anomaly equations mainly originate from the definition of the discrete indices. Second, the possibilities of forming bilinear mass terms in the presence of a non-Abelian discrete symmetry are constrained by the Kronecker products. Particles which acquire a mass at the breaking of G_f must have a $\mathcal{G}$ invariant bilinear mass term since the SU(3) irreps which are chosen to break the family gauge symmetry can only get a VEV in the direction that singles out the singlet **1** of $\mathcal{G}$. In order to discuss the effects of the massive fermionic fields on the anomaly conditions, it is therefore necessary to determine those Kronecker products which contain a singlet of the finite group.

There are two types of massive fermionic fields, Majorana particles and Dirac particles. As Majorana particles are necessarily neutral under any U(1) they don't contribute to eq. (6.1). On the other hand, the Dirac degrees of freedom always come in pairs with the two fields having opposite hypercharge; therefore their contribution to eq. (6.1) is

$$\widetilde{\ell}_{i_1}\, Y_{i_1} + \widetilde{\ell}_{i_2}\, Y_{i_2} \;=\; (\widetilde{\ell}_{i_1} - \widetilde{\ell}_{i_2})\, Y_{i_1} \;.$$

Below we evaluate this term as well as the contribution of the massive fields to eq. (6.2) explicitly for the various finite groups in turn.

- $\mathcal{PSL}_{\mathbf{2}}(\mathbf{7})$: The Kronecker products [14] show that we obtain invariant bilinear terms from the products $\mathbf{3}\otimes\overline{\mathbf{3}}$, $\mathbf{6}\otimes\mathbf{6}$, $\mathbf{7}\otimes\mathbf{7}$, $\mathbf{8}\otimes\mathbf{8}$. Applying the discrete indices defined in table 3(a), there is no contribution of massive fields to eq. (6.1), and only the sextet yields a non-zero contribution to eq. (6.2). As the **6** is a real representation of $\mathcal{PSL}_2(7)$, it can correspond to a Majorana field. In that case, because $\widetilde{A}_{\mathbf{6}} = 7$, the value of N_A on the right-hand side of eq. (6.2) is reduced from 14 to 7. Hence we have
$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;=\; 0 \bmod 24\,, \qquad \sum_{i=\text{light}} \widetilde{A}_i \;=\; 0 \bmod 7\;. \tag{6.3}$$

- $\mathcal{Z}_{\mathbf{7}} \rtimes \mathcal{Z}_{\mathbf{3}}$: The bilinear group invariants are obtained from $\mathbf{1'}\otimes\overline{\mathbf{1'}}$ and $\mathbf{3}\otimes\overline{\mathbf{3}}$, which shows that no massive Majorana particles are allowed. Only a Dirac pair of the former type has a contribution to a discrete anomaly condition which does not automatically vanish. From table 3(b) we find that such a Dirac pair adds $(2x-1)\,Y_i$ in eq. (6.1), which vanishes modulo 3 for

$$x \;=\; \frac{1}{2} \text{ or } 2\;.$$

The massive Dirac pair then does not contribute to eq. (6.1) at all. This yields

$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;=\; 0 \bmod 3\,, \qquad \sum_{i=\text{light}} \widetilde{A}_i \;=\; 0 \bmod 7\;. \tag{6.4}$$

- $\boldsymbol{\Delta(27)}$: Here the bilinear mass terms stem from the products $\mathbf{1_{2l-1}} \otimes \mathbf{1_{2l}}$ with $l = 1,\ldots,4$ as well as $\mathbf{3} \otimes \overline{\mathbf{3}}$. Again no massive Majorana particles are possible. Choosing

$$x_{2l-1} \;=\; x_{2l} \quad \text{with} \quad \sum_{l=1}^{4} x_{2l} \;=\; 0\,, \qquad \text{and} \qquad y_{2l-1} \;=\; -\,y_{2l}\,,$$

  for the discrete indices of table 4(a), the massive Dirac particles drop out of eqs. (6.1) and (6.2), yielding

$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;=\; 0 \bmod 3\,, \qquad \sum_{i=\text{light}} \widetilde{A}_i \;=\; 0 \bmod 9\;. \tag{6.5}$$

- $\mathcal{S}_\mathbf{4}$: As mass terms can be built from $\mathbf{1}' \otimes \mathbf{1}'$, $\mathbf{2} \otimes \mathbf{2}$, and $\mathbf{3_i} \otimes \mathbf{3_i}$ massive particles can be of both Majorana as well as Dirac type. Regardless of the value of x they do not contribute to eq. (6.1). On the other hand, a heavy Majorana particle living in the irrep $\mathbf{3_2}$ adds $\widetilde{A}_{\mathbf{3_2}} = 1$ to eq. (6.2), therefore changing N_A from 2 to 1 on the right-hand side. Taking

$$y \;=\; 0 \text{ or } 1\,,$$

  all discrete cubic indices are integer, leading to no useful constraint on the light particle spectrum from eq. (6.2). Using the discrete indices listed in table 4(b) with arbitrary x, a non-trivial condition only results from eq. (6.1) which reads

$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;=\; 0 \bmod 24\;. \tag{6.6}$$

- $\mathcal{A}_\mathbf{4}$: The bilinear invariants are obtained from the products $\mathbf{1}' \otimes \overline{\mathbf{1}'}$ and $\mathbf{3} \otimes \mathbf{3}$. For

$$x \;=\; 2 \text{ or } 8\,,$$

  massive particles do not contribute to eq. (6.1). The possibility of having a heavy Majorana particle in the irrep $\mathbf{3}$ reduces N_A from 2 to 1 on the right-hand side of eq. (6.2). Irrespective of the value for y, a Dirac pair with the mass term $\mathbf{1}' \otimes \overline{\mathbf{1}'}$ does not contribute to eq. (6.2). Therefore the resulting discrete anomaly condition is non-trivial; it is equivalent to the requirement of having as many $\mathbf{1}'$ as there are $\overline{\mathbf{1}'}$ in the light particle content. Explicitly, with the indices defined in table 4(c), the two discrete anomaly conditions are given as

$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i \;=\; 0 \bmod 12\,, \qquad \sum_{i=\text{light}} \widetilde{A}_i \;=\; 0 \bmod 1\;. \tag{6.7}$$

- $\mathcal{D}_5$: The masses of heavy particles can be generated from the products $\mathbf{1}' \otimes \mathbf{1}'$ and $\mathbf{2_i} \otimes \mathbf{2_i}$. Such fields can be of either Majorana or Dirac type. They do not contribute to eq. (6.1). However, each Dirac particle adds an odd integer to eq. (6.2); allowing for heavy Majorana particles, we obtain half-odd integer contributions. Since the discrete cubic indices are multiples of $\frac{1}{2}$, no useful constraint is obtained from eq. (6.2). With the indices given in table 4(d), the discrete anomaly condition obtained from eq. (6.1) yields

$$\sum_{i=\text{light}} \widetilde{\ell}_i \, Y_i \;=\; 0 \bmod 20 \; . \tag{6.8}$$

- $\mathcal{S}_3$: The bilinear invariants can originate from $\mathbf{1}' \otimes \mathbf{1}'$ and $\mathbf{2} \otimes \mathbf{2}$. As in the case of the group $\mathcal{D}_5$, massive particles do not contribute to eq. (6.1), while their possible existence renders eq. (6.2) useless. With the indices shown in table 4(e), the non-trivial discrete anomaly condition reads

$$\sum_{i=\text{light}} \widetilde{\ell}_i \, Y_i \;=\; 0 \bmod 12 \; . \tag{6.9}$$

Eqs. (6.3)–(6.9) show that the discrete anomaly conditions on the light particle spectrum depend on the finite group $\mathcal{G}$. All are subgroups of SU(3), however, $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ can alternatively be embedded into SO(3). As the discrete quadratic indices are identical for SU(3) and SO(3), the resulting discrete $G_f - G_f - \mathrm{U}(1)_Y$ anomaly conditions are identical too. Of course, for an embedding into SO(3) no cubic anomaly exists. Interestingly, even an SU(3) origin of the finite groups $\mathcal{S}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ does not yield a discrete cubic anomaly condition. Only in the case of $\mathcal{A}_4$, the second condition of eq. (6.7) is rendered useless because $\mathcal{A}_4$ could originate in SO(3) instead of SU(3). For the sake of quick reference for flavor model builders, we summarize the discrete anomaly conditions together with the necessary discrete indices in appendix C.

We emphasize that they are obtained under the assumption of an underlying anomaly-free gauge symmetry G_f. In the case where $G_f = \mathrm{U}(1)$, it can be argued that the resulting *linear* discrete anomaly conditions are identical to the requirement that the effective instanton vertex for the SM gauge theory respect the remnant $\mathcal{Z}_N$ symmetry [54]. However, non-perturbative effects cannot explain the condition arising from the cubic anomaly $\mathrm{U}(1) - \mathrm{U}(1) - \mathrm{U}(1)$, although it carries interesting information about the necessity of fractionally charged particles. Similarly, the instanton argument can also be applied to non-Abelian discrete symmetries (see e.g. ref. [55]). It is however beyond the scope of this paper to investigate the relations between the discrete anomaly conditions of eqs. (6.3)–(6.9) and the constraints arising from the requirement that the non-perturbative processes be invariant under the respective discrete symmetry.

In the following section we apply our anomaly conditions to some existing flavor models to see whether or not their preferred non-Abelian finite symmetry can be a remnant of SU(3) or SO(3), respectively.

7. Case studies

In order to illustrate the use of our work we examine some existing models of flavor. Since the SM quarks and leptons belong to the light particle spectrum, the discrete anomaly conditions can only be evaluated if the assignment of these fermions to irreps of the finite group $\mathcal{G}$ is completely given. Depending on the model, the right-handed neutrinos ν^c might also remain massless after G_f is broken down to $\mathcal{G}$. Particularly in supersymmetric models one encounters additional fermionic degrees of freedom which in general may contribute to the discrete anomalies. Here, we restrict our study to models where this is not the case, i.e. only the SM fermions (possibly including ν^c) contribute to the discrete anomaly equations, whereas other fermions are either absent, transform trivially under $\mathcal{G}$, or have $\mathcal{G}$-invariant bilinear mass terms.[6]

We first note that the sum of the hypercharges of all quarks and leptons is zero, i.e. the SM does not have a Gravity $-$ Gravity $-$ $\mathrm{U}(1)_Y$ anomaly. Therefore the mixed discrete anomaly $\mathcal{G} - \mathcal{G} - \mathrm{U}(1)_Y$ vanishes identically if the SM fermions all live in the same representation of the finite group $\mathcal{G}$. This is the case for the models of refs. [15, 17, 21, 31]. The discrete symmetries employed in [15, 17] are $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ and $\Delta(27)$, respectively, which can only be embedded in SU(3). One therefore still has to check the discrete cubic anomaly. With the fermions transforming as triplets of $\mathcal{G}$ in both cases, we have

$$\sum_{i=\text{light}} \widetilde{A}_i \;=\; \sum_{i=\text{light}} 1 \;=\; 16\;.$$

The comparison with eqs. (6.4) and (6.5) shows that both models are not discrete anomaly free and therefore incomplete: they require additional fermions which do not acquire mass when SU(3) is broken down to the discrete family symmetry $\mathcal{G}$.

As mentioned above, the models of refs. [21, 31] have no mixed discrete anomaly $\mathcal{G} - \mathcal{G} - \mathrm{U}(1)_Y$. Since the applied family symmetries $\mathcal{S}_4$ and $\mathcal{A}_4$ are subgroups of SO(3), these models are not constrained by the cubic anomaly condition and therefore discrete anomaly free. Similarly, one has to check only the mixed discrete anomaly for the examples listed in table 6. In all models, the SM fermions are the only particles contributing to the discrete anomaly, which is therefore determined solely by the assignment of the quarks and leptons to irreps of $\mathcal{G}$. Whenever the value for the discrete anomaly, given in the rightmost column, is non-zero, the model is not discrete anomaly free, i.e. it is necessary to include additional light fermions or, alternatively, heavy fermions with fractional hypercharges.

8. Conclusion

In recent years, many flavor models invoking the operation of a non-Abelian discrete family symmetry have been suggested to explain the tri-bimaximal leptonic mixing pattern. This plethora of possibilities asks for criteria to assess the viability of a model. In our study we have formulated the consequences of embedding non-Abelian discrete symmetries $\mathcal{G}$ into a continuous gauge symmetry G_f. Mathematical consistency requires the underlying gauge

[6]For $\mathcal{G} \subset \mathrm{SO}(3)$, light fermions with zero hypercharge do not enter the anomaly equation either.

Group	refs.	Q	u^c	d^c	L	e^c	$\sum \widetilde{\ell}_i Y_i$
$\mathcal{A}_4$	[24, 25, 29]	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$0 \bmod 12$
$\mathcal{A}_4$	[24]	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$6 \bmod 12$
$\mathcal{A}_4$	[24, 26, 29]	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$6 \bmod 12$
$\mathcal{A}_4$	[30]	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{3}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$0 \bmod 12$
$\mathcal{A}_4$	[32]	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}', \overline{\mathbf{1}'}$	$\mathbf{3}$	$6 \bmod 12$
$\mathcal{D}_5$	[37]	$\mathbf{1}, \mathbf{2_2}$	$\mathbf{1}, \mathbf{2_1}$	$\mathbf{1}, \mathbf{2_1}$	$\mathbf{1}, \mathbf{2_2}$	$\mathbf{1}, \mathbf{2_1}$	$0 \bmod 20$
$\mathcal{S}_3$	[44]	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{2}$	$\mathbf{1}, \mathbf{2}$	$0 \bmod 12$
$\mathcal{S}_3$	[45]	$\mathbf{1}, \mathbf{2}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{1}, \mathbf{1}$	$\mathbf{1}, \mathbf{2}$	$\mathbf{1}', \mathbf{1}', \mathbf{1}'$	$3 \bmod 12$

Table 6: With the particle content given in these existing flavor models, only the SM fermions contribute to the mixed discrete anomaly. Thus the assignment of the quarks and leptons to irreps of a finite group determines whether a model is discrete anomaly free.

theory to be anomaly free; this translates to discrete anomaly conditions after the breaking of G_f. A model builder's toolbox for quickly checking the discrete anomaly conditions of a model is provided for in appendix C.

Acknowledgments

We are indebted to G. Ross for stimulating discussions during the initial stages of this work. We also wish to thank Y. Tachikawa for his inductive proof for the cubic index of $\mathcal{PSL}_2(7)$. PR acknowledges support from the Institute for Advanced Study, the Ambrose Monell foundation, and the Department of Energy Grant No. DE-FG02-97ER41029. CL is supported by the University of Florida through the Institute for Fundamental Theory.

A. Obtaining all SU(3) representations successively

It is well known that the irreps of SU(3) can be constructed from solely the fundamental triplet. Still, in the inductive step of our proof of eq. (2.6) we check the validity of this equation not only for $\boldsymbol{\rho} \otimes \mathbf{3}$ but also for $\boldsymbol{\rho} \otimes \overline{\mathbf{3}}$. Considering the $\overline{\mathbf{3}}$ as well can potentially add new constraints to the definition of the discrete indices. For example, without the $\overline{\mathbf{3}}$, the first table of appendix B suggests that the discrete cubic indices of $\mathcal{PSL}_2(7)$ could be defined modulo $N_A = 28$; however, the lower half of the table reveals that, in fact, we need $N_A = 14$.

To better understand the reason for why the $\overline{\mathbf{3}}$ must be included in our proof, let us discuss all possible ways for obtaining the $\mathbf{15}$ of SU(3) by multiplying the $\mathbf{3}$ with some

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smaller irrep for which eq. (2.6) shall hold. From the Young tableau of the **15**

$$\ydiagram{3,1} \ ,$$

it is obvious that there are only two smaller irreps which, after multiplication with the **3**, include the **15**. The corresponding products are

$$\ydiagram{2,1} \otimes \ydiagram{1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{1} \ ,$$
$$\ydiagram{3} \otimes \ydiagram{1} = \ydiagram{3,1} \oplus \ydiagram{4} \ .$$

In both cases, we obtain the **15** *and another new* irrep for which the validity of eq. (2.6) has not been shown.[7] Therefore, with multiplications by $\boldsymbol{\sigma} = \mathbf{3}$, we can only prove that eq. (2.6) holds true for *a sum of two new* irreps.

This shortcoming can be overcome by adding the choice of $\boldsymbol{\sigma} = \overline{\mathbf{3}}$. Then, all irreps can be successively generated with *only one new* irrep appearing on the right-hand side of the corresponding products. Assume that we knew all irreps of the form

$$\begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k \end{ytableau} \ , \qquad \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k \\ 1 \end{ytableau} \ , \tag{A.1}$$

with $1 \leq k \leq K$. Notice that the case with $k = 1$ comprises the basis of our proof by induction; hence we must initially show that eq. (2.6) is true for both, the **3** and the $\overline{\mathbf{3}}$. We now determine the two products

$$\begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & K \end{ytableau} \otimes \ydiagram{1} = \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & K & \none[\ ] \end{ytableau}\!\!\!\!\ydiagram{1} \oplus \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & K \\ \ \end{ytableau} \ , \tag{A.2}$$

$$\begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & K \end{ytableau} \otimes \ydiagram{1,1} = \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & K & \ \\ \ \end{ytableau} \oplus \begin{ytableau} 2 & \cdot & \cdot & \cdot & K \end{ytableau} \ . \tag{A.3}$$

The second irreps on the right-hand sides of eqs. (A.2) and (A.3) are already known. The first ones are new and extend eq. (A.1) to $k \leq K+1$. This shows that the irreps in eq. (A.1) can be obtained with arbitrary $k \in \mathbb{N}$.

We can now fill up the second row of the Young tableaux by multiplications with **3**. Assume that the irreps of the form

$$\begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k \\ 1 & 2 & \cdot & l \end{ytableau} \ , \tag{A.4}$$

are known for arbitrary $k \in \mathbb{N}$ and $1 \leq l \leq L$. Notice that the case $L = 1$ is nothing but the second Young tableau of eq. (A.1). Then

$$\begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k \\ 1 & 2 & \cdot & L \end{ytableau} \otimes \ydiagram{1} = \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k \\ 1 & 2 & \cdot & L & \ \end{ytableau} \oplus \begin{ytableau} 1 & 2 & \cdot & \cdot & \cdot & k & \ \\ 1 & 2 & \cdot & L \end{ytableau} \oplus \begin{ytableau} 2 & \cdot & \cdot & \cdot & k \\ 2 & \cdot & L \end{ytableau} \ .$$

[7]One might have the impression that the $\overline{\mathbf{6}}$ in the first product should *automatically* satisfy eq. (2.6). This however is incorrect since, for the cubic index, we cannot infer that $-A(\mathbf{6}) = A(\overline{\mathbf{6}}) = \widetilde{A}(\mathbf{6}) \bmod N_A$ from the validity of $A(\mathbf{6}) = \widetilde{A}(\mathbf{6}) \bmod N_A$ only. One must prove eq. (2.6) separately for the $\overline{\mathbf{6}}$.

Only the first irrep on the right-hand side is a new one and extends eq. (A.4) to $l \leq L+1$, and thus to arbitrary $l \in \mathbb{N}$. Therefore any irrep of SU(3) can be generated successively by multiplication with $\mathbf{3}$ and $\overline{\mathbf{3}}$ in a way that only one new irrep occurs on the right-hand side of the tensor product.[8]

B. The proof of eq. (2.11) for $\mathcal{PSL}_2(7)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$

This appendix shows the explicit values of $f_I^i(\boldsymbol{\sigma})$ and $\mathfrak{f}_I^i(\boldsymbol{\sigma})$ for the finite groups $\mathcal{PSL}_2(7)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$. They are calculated from eqs. (3.2)–(3.4) with the discrete indices given in table 3. The comparison proves that eq. (2.11) is satisfied for all i. Therefore our definition of the discrete indices is consistent.

$\mathcal{PSL}_2(7)$	Quadratic Index ℓ ($N_\ell = 24$)		Cubic Index A ($N_A = 14$)	
i	$f_\ell^i(\mathbf{3})$	$\mathfrak{f}_\ell^i(\mathbf{3})$	$f_A^i(\mathbf{3})$	$\mathfrak{f}_A^i(\mathbf{3})$
0	$1+0=1$	$1=1$	$1+0=1$	$1=1$
1	$3+3=6$	$1+5=6$	$3+3=6$	$-1+7=6$
2	$3+3=6$	$0+6=6$	$3-3=0$	$0+0=0$
3	$6+15=21$	$1+14+6=21$	$6+21=27$	$-1+0+0=-1$
4	$7+42=49$	$5+14+6=25$	$7+0=7$	$7+0+0=7$
5	$8+18=26$	$1+5+14+6=26$	$8+0=8$	$1+7+0+0=8$
i	$f_\ell^i(\overline{\mathbf{3}})$	$\mathfrak{f}_\ell^i(\overline{\mathbf{3}})$	$f_A^i(\overline{\mathbf{3}})$	$\mathfrak{f}_A^i(\overline{\mathbf{3}})$
0	$1+0=1$	$1=1$	$-1+0=-1$	$-1=-1$
1	$3+3=6$	$0+6=6$	$-3+3=0$	$0+0=0$
2	$3+3=6$	$1+5=6$	$-3-3=-6$	$1+7=8$
3	$6+15=21$	$1+14+6=21$	$-6+21=15$	$1+0+0=1$
4	$7+42=49$	$5+14+6=25$	$-7+0=-7$	$7+0+0=7$
5	$8+18=26$	$1+5+14+6=26$	$-8+0=-8$	$-1+7+0+0=6$

$\mathcal{Z}_7 \rtimes \mathcal{Z}_3$	Quadratic Index ℓ ($N_\ell = 3$)		Cubic Index A ($N_A = 7$)	
i	$f_\ell^i(\mathbf{3})$	$\mathfrak{f}_\ell^i(\mathbf{3})$	$f_A^i(\mathbf{3})$	$\mathfrak{f}_A^i(\mathbf{3})$
0	1	1	1	1
$1+2$	5	2	2	2
3	6	3	6	-1
4	6	3	0	0
i	$f_\ell^i(\overline{\mathbf{3}})$	$\mathfrak{f}_\ell^i(\overline{\mathbf{3}})$	$f_A^i(\overline{\mathbf{3}})$	$\mathfrak{f}_A^i(\overline{\mathbf{3}})$
0	1	1	-1	-1
$1+2$	5	2	-2	-2
3	6	3	0	0
4	6	3	-6	1

[8]It is worth mentioning that this method can be generalized to Lie groups other than SU(3). Then $\boldsymbol{\sigma}$ has to take all irreps associated with the fundamental weights: for example, in SU(4) these are $\mathbf{4}$, $\mathbf{6}$, $\overline{\mathbf{4}}$. Thanks to Dr. Yuji Tachikawa for pointing this out.

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$\mathcal{G} \subset \mathrm{SU}(3)$	$\mathcal{PSL}_2(7)$	$\mathcal{Z}_7 \rtimes \mathcal{Z}_3$	$\Delta(27)$
$\widetilde{\ell}(\mathbf{r_i})$	$\mathbf{3}:\ 1$ $\overline{\mathbf{3}}:\ 1$ $\mathbf{6}:\ 5$ $\mathbf{7}:\ 14$ $\mathbf{8}:\ 6$	$\mathbf{1'}:\ 1/2 \quad 2$ $\overline{\mathbf{1'}}:\ 1/2 \quad 2$ $\mathbf{3}:\ 1 \quad 1$ $\overline{\mathbf{3}}:\ 1 \quad 1$	$\mathbf{1_{2l}}:\ x_{2l}$ $\mathbf{1_{2l-1}}:\ x_{2l}$ $\mathbf{3}:\ 1$ $\overline{\mathbf{3}}:\ 1$
$\sum_{i=\text{light}} \widetilde{\ell}_i Y_i$	0 mod 24	0 mod 3	0 mod 3
$\widetilde{A}(\mathbf{r_i})$	$\mathbf{3}:\ 1$ $\overline{\mathbf{3}}:\ -1$ $\mathbf{6}:\ 0$ $\mathbf{7}:\ 0$ $\mathbf{8}:\ 0$	$\mathbf{1'}:\ y$ $\overline{\mathbf{1'}}:\ -y$ $\mathbf{3}:\ 1$ $\overline{\mathbf{3}}:\ -1$	$\mathbf{1_{2l}}:\ y_{2l}$ $\mathbf{1_{2l-1}}:\ -y_{2l}$ $\mathbf{3}:\ 1$ $\overline{\mathbf{3}}:\ -1$
$\sum_{i=\text{light}} \widetilde{A}_i$	0 mod 7	0 mod 7	0 mod 9

Table 7: The discrete indices for the irreps of $\mathcal{PSL}_2(7)$, $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, and $\Delta(27)$ together with the corresponding discrete anomaly conditions.

$\mathcal{G} \subset \mathrm{SO}(3)$	$\mathcal{S}_4$	$\mathcal{A}_4$	$\mathcal{D}_5$	$\mathcal{S}_3$
$\widetilde{\ell}(\mathbf{r_i})$	$\mathbf{1'}:\ 13-x$ $\mathbf{2}:\ 5-x$ $\mathbf{3_1}:\ x$ $\mathbf{3_2}:\ 1$	$\mathbf{1'}:\ 2 \quad 8$ $\overline{\mathbf{1'}}:\ 2 \quad 8$ $\mathbf{3}:\ 1 \quad 1$	$\mathbf{1'}:\ 5/2 \quad 25/2$ $\mathbf{2_1}:\ -3/2 \quad 17/2$ $\mathbf{2_2}:\ 13/2 \quad 33/2$	$\mathbf{1'}:\ -3/2 \quad 9/2$ $\mathbf{2}:\ 5/2 \quad 17/2$
$\sum_{i=\text{light}} \widetilde{\ell}_i Y_i$	0 mod 24	0 mod 12	0 mod 20	0 mod 12

Table 8: The discrete indices for the irreps of $\mathcal{S}_4$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ together with the corresponding discrete anomaly conditions.

C. Toolbox for model builders

In this appendix we collect all the results of our study which are relevant for flavor model building. We tabulate the discrete indices for each finite group, now taking into account the constraints from the mass terms discussed in section 6. Thus some of the parameters of tables 3 and 4 are fixed. Others remain undetermined and the discrete anomaly conditions must be satisfied for arbitrary values. Only in the case of $\Delta(27)$ the parameters x_{2l} with $l = 1, \ldots, 4$ are additionally constrained by the condition $\sum_{l=1}^{4} x_{2l} = 0$. For the groups $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, $\mathcal{A}_4$, $\mathcal{D}_5$, and $\mathcal{S}_3$ there are two inequivalent ways to assign discrete quadratic indices. Therefore, anomaly freedom of the underlying continuous family symmetry G_f requires that the anomaly conditions be satisfied for *both* of these choices separately. When calculating the discrete anomaly $\mathcal{G}-\mathcal{G}-\mathrm{U}(1)_Y$ it is necessary to choose the normalization with $Y_Q = 1$.

Table 7 shows the finite groups which are subgroups of SU(3) only. Therefore the discrete cubic anomaly provides a useful condition. In table 8 we list the finite groups which can be considered as subgroups of SO(3) as well. Hence, the discrete cubic anomaly condition of $\mathcal{A}_4 \subset \mathrm{SU}(3)$ is omitted, see section 6.

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